Limite de funcții I. Peter Radu

Saturday, March 13, 2021 9:03 AM

**321.** $\lim\limits_{x\to\infty} \dfrac{(x+\sqrt{2})^{\sqrt{2}}-(x-\sqrt{2})^{\sqrt{2}}}{x^{\sqrt{2}-1}}$

$a, b, \alpha \in \mathbb{R} \quad \alpha > 0$

$\lim\limits_{x\to\infty} \dfrac{(x+a)^{\alpha}-(x+b)^{\alpha}}{x^{\alpha-1}}$

$g:[a,b]\to\mathbb{R} \quad g(t)=(x+t)^{\alpha} \to t$ - variabilă

cont pe $[a,b]$, derivabilă pe $(a,b)$

T. Lagrange: $\exists\, c\in(a,b)$ a.î. $g(a)-g(b)=(a-b)\,g'(c)$, $c\in(a,b)$

$\exists\, c\in(a,b) \quad (x+a)^{\alpha}-(x+b)^{\alpha}=(a-b)\,\alpha\cdot(x+c)^{\alpha-1}$

$\dfrac{(x+c)^{\alpha}-(x+b)^{\alpha}}{x^{\alpha-1}}=(a-b)\alpha\dfrac{(x+c)^{\alpha-1}}{x^{\alpha-1}} \qquad c\in(a,b)$

$\dfrac{(x+a)^{\alpha}-(x+b)^{\alpha}}{x^{\alpha-1}}=(a-b)\alpha\left(\dfrac{x+c}{x}\right)^{\alpha-1}$

$\lim\limits_{x\to\infty}\left(\dfrac{x+c}{x}\right)^{\alpha-1}=1$

$\lim\limits_{x\to\infty}\dfrac{(x+a)^{\alpha}-(x+b)^{\alpha}}{x^{\alpha-1}}=\alpha(a-b)\lim\limits_{x\to\infty}\left(\dfrac{x+c}{x}\right)^{\alpha-1}=\alpha(a-b)$

$\lim\limits_{x\to\infty}\dfrac{(x+\sqrt{2})^{\sqrt{2}}-(x-\sqrt{2})^{\sqrt{2}}}{x^{\sqrt{2}-1}}=\sqrt{2}\cdot 2\sqrt{2}=4$

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**322** $\lim\limits_{x\to 0}\dfrac{(1+ax)^{1/x}-(1+x)^{\frac{a}{x}}}{x} \qquad a\in\mathbb{R}$

$g(x)=(1+x)^{\frac{a}{x}} \to g(ax)=(1+ax)^{1/x}$

$\dfrac{(1+ax)^{1/x}-(1+x)^{\frac{a}{x}}}{x}=\dfrac{g(ax)-g(x)}{x}$

$g:(0,\infty)\to\mathbb{R} \quad g(x)=(1+x)^{\frac{a}{x}}$ cont, derivabilă

$\exists$ c între x și ax a.î.

$g(ax)-g(x)=(ax-x)\cdot g'(c) \to$

$\dfrac{g(ax)-g(x)}{x}=\dfrac{ax-x}{x}\cdot g'(c)$

$\dfrac{g(ax)-g(x)}{x}=(a-1)\cdot g'(c) \qquad g(c)=(1+c)^{\frac{a}{c}}=u^{v}=e^{v\ln u}$

$g'(c)=e^{v\ln u}\left[v'\ln u+v\cdot\dfrac{u'}{u}\right]$

$x\to 0 \iff c\to 0$ (c este între x și ax) !!

$g'(c)=(1+c)^{\frac{a}{c}}\left[-\dfrac{a}{c^2}\cdot\ln(1+c)+\dfrac{a}{c}\cdot\dfrac{1}{1+c}\right]=$

$=(1+c)^{\frac{a}{c}}\,a\,\dfrac{c-(1+c)\cdot\ln(1+c)}{c^2(1+c)}$

$\lim\limits_{x\to 0}\dfrac{(1+ax)^{1/x}-(1+x)^{a/x}}{x}=(a-1)\,a\lim\limits_{c\to 0}\underbrace{(1+c)^{a/c}}_{e^{a}}\cdot\dfrac{c-(1+c)\ln(1+c)}{c^2(1+c)}$

$\lim\limits_{c\to 0}\dfrac{c-(1+c)\cdot\ln(1+c)}{c^2}\overset{l'H}{=}$

$$\lim_{c\to 0} \frac{c-(1+c)\cdot \ln(1+c)}{c^2} \overset{l'H}{=}$$

$$= \lim_{c\to 0} \frac{1-\ln(1+c)-(1+c)\cdot\frac{1}{1+c}}{2c} \overset{l'H}{=} \lim_{c\to 0}\left(-\frac{\ln(1+c)}{2c}\right) \overset{l'H}{=} -\lim \frac{1}{2(1+c)} = -\frac{1}{2}$$

$$\lim_{c\to 0} g'(c) = e^a\cdot\left(-\frac{1}{2}\right) = -\frac{e^a}{2}$$

$$\boxed{\lim_{x\to 0} \frac{(1+ax)^{1/x}-(1+x)^{a/x}}{x} = \frac{a(1-a)}{2}\cdot e^a}$$

$g:(0,\infty)\to\mathbb{R}$ $\quad \lim_{x\to 0} g'(c) = L$

$$\lim_{x\to 0}\frac{g(ax)-g(x)}{x} =$$

$$\frac{g(ax)-g(x)}{x} = \frac{(ax-x)}{x}\cdot g'(c)$$

$c$ - între $x$ și $ax$

$$=(a-1)\cdot g'(c)$$

$x\to 0 \iff c\to 0$

$$\lim_{x\to 0}\frac{g(ax)-g(x)}{x} = (a-1)L$$

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(320) $$\lim_{x\to 0}\frac{x-\overbrace{\sin(\sin\dots(\sin x))}^{n \text{ ori}}}{x^3} = L_n$$

$$L_{n-1} = \lim_{x\to 0}\frac{x-\overbrace{\sin(\sin\dots(\sin x)\dots)}^{n-1}}{x^3}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

$$\frac{x-\overbrace{\sin(\sin\dots(\sin x)\dots)}^{n \text{ ori}}}{x^3} = \frac{x-\sin x+\sin x-\sin(\sin\dots(\sin x)\dots)}{x^3}$$

$$\frac{x-\overbrace{\sin(\sin\dots(\sin x))}^{n \text{ ori}}}{x^3} = \frac{x-\sin x}{x^3} + \frac{\sin x - \overbrace{\sin\dots(\sin x)}^{n \text{ ori}}}{\sin^3 x}\cdot\frac{\sin^3 x}{x^3}$$

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} \overset{l'H}{=} \lim_{x\to 0}\frac{1-\cos x}{3x^2} \overset{l'H}{=} \lim \frac{\sin x}{6x} = \frac{1}{6}$$

$$\lim_{x\to 0}\frac{\sin x-\overbrace{\sin(\sin\dots(\sin x)\dots)}^{n \text{ ori}}}{\sin^3 x} = (*)$$

$$\sin x = t \quad (x\to 0 \iff t\to 0)$$

$$(*) = \lim_{t\to 0}\frac{t-\overbrace{\sin\dots(\sin t)}^{(n-1) \text{ ori}}}{t^3} = L_{n-1}$$

$$\lim_{x\to 0}\frac{x-\sin(\sin\dots(\sin x)\dots)}{x^3} = \frac{1}{6} + \lim_{x\to 0}\frac{x-\sin(\sin\dots(\sin x)\dots)}{x^3}.$$

$$L_n = \frac{1}{6} + L_{n-1}$$

$$L_1 = \lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6} \to L_2 \exists \dots \quad L_n \exists \ \forall n\in\mathbb{N}$$

$$L_n = \frac{n}{6}$$

$$L_n = \frac{n}{6}$$

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(335) $$\lim_{x\to 0} \frac{\sin x^n - \sin^n x}{x^{n+2}} \qquad n \in \mathbb{N},\ n > 2$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad\qquad \lim_{x\to 0} \frac{x^n - \sin x^n}{x^{3n}} = \frac{1}{6}$$

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$$

$$\frac{\sin x^n - x^n + x^n - \sin^n x}{x^{n+2}} = \underbrace{\frac{\sin x^n - x^n}{x^{3n}}}_{\downarrow\, 1/6} \cdot \underbrace{\frac{x^{3n}}{x^{n+2}}}_{x^{2n-2}} + \overbrace{\frac{x^n - \sin^n x}{x^{n+2}}}^{n/6}$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2} b + a^{n-3} b^2 + \dots + a \cdot b^{n-2} + b^{n-1})$$

$$x^n - \sin^n x = (x - \sin x)(\sin^{n-1} x + x \sin^{n-2} x + x^2 \sin^{n-3} x + \dots + x^{n-2} \cdot \sin x + x^{n-1})$$

$$\frac{x^n - \sin^n x}{x^{n+2}} = \frac{x - \sin x}{x^3} \cdot \frac{\sin^{n-1} x + x \sin^{n-2} x + \dots + x^{n-2} \cdot \sin x + x^{n-1}}{x^{n-1}} =$$

$$= \frac{x - \sin x}{x^3} \left( \left(\frac{\sin x}{x}\right)^{n-1} + \left(\frac{\sin x}{x}\right)^{n-2} + \dots + \frac{\sin x}{x} + 1 \right)$$

$$\lim_{x\to 0} \frac{x^n - \sin^n x}{x^{n+2}} = \lim_{x\to 0} \frac{x - \sin x}{x^3} \cdot \lim_{x\to 0} \left( \left(\frac{\sin x}{x}\right)^{n-1} + \left(\frac{\sin x}{x}\right)^{n-2} + \dots + \frac{\sin x}{x} + 1 \right)$$

$$= \frac{n}{6}$$

$$\lim_{x\to 0} \frac{\sin x^n - \sin^n x}{x^{n+2}} = \lim_{x\to 0} \frac{\sin x^n - x^n}{x^{3n}} \cdot x^{2n-2} + \lim_{x\to 0} \frac{x^n - \sin^n x}{x^{n+2}}$$

$$= \frac{n}{6}$$

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(531) $$\lim_{t\to 0} \int_{2^t}^{3^t} \frac{x}{\ln x}\, dx \qquad x \qquad 2^t \longrightarrow 3^t$$

$$y \qquad x = y^t \qquad y \in [2, 3]$$

$$\ln x = t \ln y \qquad dx = t \cdot y^{t-1} dy$$

$$\int \frac{dy}{y \ln y} = \ln \ln y$$

$$\int_{2^t}^{3^t} \frac{x}{\ln x}\, dx = \int_2^3 \frac{y^t}{\cancel{t} \ln y} \cancel{t} \cdot y^{t-1}\, dy =$$

$$\int_2^3 \frac{dy}{y \ln y} =$$

$$= \int_2^3 y^{2t-1} \cdot \frac{1}{\ln y}\, dy = \underbrace{\int_2^3 y^{2t} \frac{1}{y \ln y}\, dy}_{I_t}$$

$$= \ln \ln 3 - \ln \ln 2 = \ln\left(\frac{\ln 3}{\ln 2}\right)$$

$$\int_2^3 2^t \frac{1}{y \ln y}\, dy \le \underbrace{\int_2^3 y^{2t} \boxed{\frac{1}{y \ln y}}\, dy}_{} \le \int 3^t \frac{1}{y \ln y}\, dy$$

$$2t \int^3 \dots \le \dots 2t \dots \qquad \boxed{\lim \int^3 y^{2t} \frac{1}{\dots}\, dy = ?}$$

$$2^{2t}\cdot\cancel{1}\int_2^3 \frac{1}{y\ln y}\,dy \le I_t \le 3^{2t}\cdot\cancel{1}\int \frac{1}{y\ln y}\,dy$$

$$\boxed{\lim_{t\to 0}\int_2^3 y^{2t}\frac{1}{y\ln y}\,dy \overset{?}{=} \int_2^3 \lim_{t\to 0} y^{2t}\cdot\frac{1}{y\ln y}\,dy}$$

$$2^{2t}\cdot\ln\frac{\ln 3}{\ln 2} \le I_t \le 3^{2t}\ln\frac{\ln 3}{\ln 2}$$

$t\to 0$

$$\ln\frac{\ln 3}{\ln 2} \Rightarrow \lim_{t\to 0}\int_{2^t}^{3^t}\frac{x}{\ln x}\,dx = \ln\frac{\ln 3}{\ln 2}$$